Week 13 Discussion

Tuesday, April 20, 2021 10:18 PM

6.1

Problem 1. *Use the Gaussian Elimination Algorithm to solve the following linear systems, if possibly, and determine whether row interchanges are necessary:*

$$\begin{cases} x - y + 3z &= 2 \\ 3x - 3y + z &= 1 \\ x + y &= 3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 3 & -3 & 1 & 1 \\ 1 & 1 & 0 & 3 \end{array}\right]$$

$r_2 \to r_2 - 3 \cdot r_1$

$r_3 \to r_3 - r_1$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 0 & -8 & -5 \\ 0 & 2 & -3 & -2 \end{array}\right]$$

$r_1 \to r_1 + r_3$

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 5 \\ 3 & -3 & 1 & 1 \\ 1 & 1 & 0 & 3 \end{array}\right]$$

$r_2 \to r_2 - \frac{3}{2} r_1$

$r_3 \to r_3 - \frac{1}{2} r_1$

$$\left(\begin{array}{ccc|c} 2 & 0 & 3 & 5 \\ 0 & -3 & -5/4 & -2 \\ 0 & 1 & -3/2 & -1 \end{array}\right)$$

$r_3 \to r_3 + \frac{1}{3} r_2$

$$\left(\begin{array}{ccc|c} 2 & 0 & 3 & 5 \\ 0 & -3 & -5/4 & -2 \\ 0 & 0 & -8/5 & -4/5 \end{array}\right)$$

Problem 2. *Given the linear system*

$$\begin{cases} x - y + \alpha z &= -2 \\ -x + 2y - \alpha z &= 3 \\ \alpha x + y + z &= 2 \end{cases}$$

1. *find the values of α for which the system has no solutions.*
2. *find the values of α for which the system has an infinite number of solutions.*
3. *assuming a unique solution exists for a given α, find the solution.*

$$\begin{bmatrix} 1 & -1 & \alpha \\ -1 & 2 & -\alpha \\ \alpha & 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$$

$r_2 \to r_2 + r_1$

$r_3 \to r_3 - \alpha r_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & \alpha & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 1+\alpha & 1-\alpha^2 & 2+2\alpha \end{array}\right]$$

$-\alpha r_1 = (-\alpha \quad +\alpha \quad -\alpha^2 \quad +2\alpha)$

$y = 1$

$(1+\alpha) + (1-\alpha^2) z = 2 + 2\alpha$

$(1-\alpha^2) z = 1 + \alpha$

$[(1+\alpha)(1-\alpha) z = 1+\alpha]$

if $\boxed{\alpha = -1}$ prob. infinite

$\alpha = -1$

z anything.

$$\boxed{\begin{aligned} x &= -1 - \alpha z \\ y &= 1 \\ z &\in \mathbb{R} \end{aligned}}$$

if $\alpha \neq -1$

$(1-\alpha) z = 1$

$z = \frac{1}{1-\alpha}$

$\boxed{\alpha \neq 1}$

$x - 1 + \frac{\alpha}{1-\alpha} = -2$

$$\boxed{\begin{aligned} x &= -1 - \frac{\alpha}{1-\alpha} \\ z &= \frac{1}{1-\alpha} \\ y &= 1 \end{aligned}}$$

$\alpha = 1$

no solution.

6.2

Problem 3. *Find the row interchanges that are required to solve the following linear system using Algorithm 6.1*

$$\begin{cases} x - 5y + z &= 7 \\ 10x + 20z &= 6 \\ 5x - z &= 4 \end{cases}$$

2

6.3

Problem 4. *Is the following matrix nonsingular? If it is compute the inverse:*

$$\begin{pmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\hphantom{4 \quad 2 \quad 6}}^{A}} & & & \\ 4 & 2 & 6 & 1 & 0 & 0 \\ 3 & 0 & 7 & 0 & 1 & 0 \\ -2 & 0 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right]$$

Problem 5. *Prove the following:*

1. *if A^{-1} exists, it is unique*
2. *if A is nonsingular, then $(A^{-1})^{-1} = A$*
3. *If A and B are nonsingular matricies of the same size, then $(AB)^{-1} = B^{-1}A^{-1}$*

1) Let B & $\tilde{B}$ be inverses of A

$$B = B \cdot I = B(A\tilde{B}) = (BA)\tilde{B} = I\tilde{B} = \tilde{B}$$

$$\Rightarrow B = \tilde{B}$$

2) Let $\boxed{B = A^{-1}}$

Want to show that $B^{-1} = A$

ie $AB = BA = I$

This holds because B is the inverse of A.

3) WTS $(AB)^{-1} = B^{-1}A^{-1}$

Prove $(B^{-1}A^{-1})(AB) = I$ ✓

3) WTS $(AB)^{-1} = B^{-1}A^{-1}$

ie prove $(B^{-1}A^{-1})(AB) = I$ ✓

& $(AB)(B^{-1}A^{-1}) = I$ ✓

$$(B^{-1}A^{-1})(AB) = B^{-1}(\underbrace{A^{-1}A}_{})B$$
$$= B^{-1} I B$$
$$= B^{-1}(IB)$$
$$= B^{-1}B$$
$$= I$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$
$$= A(I)A^{-1}$$
$$= AA^{-1} = I$$